MAA
MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

American Mathematics Competitions

64th Annual

AMC 12 A

American Mathematics Contest 12 A
Tuesday, February 5, 2013

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, email, internet or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:

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The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:

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1. **Answer (E):** The legs of $\triangle ABE$ have lengths $AB = 10$ and BE. Therefore $\frac{1}{2} \cdot 10 \cdot BE = 40$, so $BE = 8$.

2. **Answer (C):** The softball team could only have scored twice as many runs as their opponent when they scored an even number of runs. In those games their opponents scored

$$\frac{2}{2} + \frac{4}{2} + \frac{6}{2} + \frac{8}{2} + \frac{10}{2} = 15 \text{ runs.}$$

In the games the softball team lost, their opponents scored

$$(1+1) + (3+1) + (5+1) + (7+1) + (9+1) = 30 \text{ runs.}$$

The total number of runs scored by their opponents was $15 + 30 = 45$ runs.

3. **Answer (E):** Because six tenths of the flowers are pink and two thirds of the pink flowers are carnations, $\frac{6}{10} \cdot \frac{2}{3} = \frac{2}{5}$ of the flowers are pink carnations. Because four tenths of the flowers are red and three fourths of the red flowers are carnations, $\frac{4}{10} \cdot \frac{3}{4} = \frac{3}{10}$ of the flowers are red carnations. Therefore $\frac{2}{5} + \frac{3}{10} = \frac{7}{10} = 70\%$ of the flowers are carnations.

4. **Answer (C):** Factoring 2^{2012} from each of the terms and simplifying gives

$$\frac{2^{2012}(2^2+1)}{2^{2012}(2^2-1)} = \frac{4+1}{4-1} = \frac{5}{3}.$$

5. **Answer (B):** The total shared expenses were $105+125+175 = 405$ dollars, so each traveler's fair share was $\frac{1}{3} \cdot 405 = 135$ dollars. Therefore $t = 135 - 105 = 30$ and $d = 135 - 125 = 10$, so $t - d = 30 - 10 = 20$.

OR

Because Dorothy paid 20 dollars more than Tom, Sammy must receive 20 more dollars from Tom than from Dorothy.

6. **Answer (B):** If Shenille attempted x three-point shots and $30 - x$ two-point shots, then she scored a total of $\frac{20}{100} \cdot 3 \cdot x + \frac{30}{100} \cdot 2 \cdot (30 - x) = 18$ points.

Remark: The given information does not allow the value of x to be determined.

7. **Answer (C):**

   Note that $110 = S_9 = S_7 + S_8 = 42 + S_8$, so $S_8 = 110 - 42 = 68$. Thus $68 = S_8 = S_6 + S_7 = S_6 + 42$, so $S_6 = 68 - 42 = 26$. Similarly, $S_5 = 42 - 26 = 16$, and $S_4 = 26 - 16 = 10$.

8. **Answer (D):**

   Multiplying the given equation by $xy \neq 0$ yields $x^2y + 2y = xy^2 + 2x$. Thus

   $$x^2y - 2x - xy^2 + 2y = x(xy - 2) - y(xy - 2) = (x - y)(xy - 2) = 0.$$

   Because $x - y \neq 0$, it follows that $xy = 2$.

9. **Answer (C):** Because $\overline{EF}$ is parallel to $\overline{AB}$, it follows that $\triangle FEC$ is similar to $\triangle ABC$ and $FE = FC$. Thus half of the perimeter of $ADEF$ is $AF + FE = AF + FC = AC = 28$. The entire perimeter is 56.

10. **Answer (D):** If n satisfies the equation $\frac{1}{n} = 0.\overline{ab}$, then $\frac{100}{n} = ab.\overline{ab}$ and subtracting gives $\frac{99}{n} = ab$. The positive factors of 99 are 1, 3, 9, 11, 33, and 99. Only $n = 11$, 33, and 99 give a number $\frac{99}{n}$ consisting of two different digits, namely 09, 03, and 01, respectively. Thus the requested sum is $11+33+99 = 143$.

11. **Answer (C):**

    Let $x = DE$ and $y = FG$. Then the perimeter of ADE is $x + x + x = 3x$, the perimeter of $DFGE$ is $x + (y - x) + y + (y - x) = 3y - x$, and the perimeter of $FBCG$ is $y + (1 - y) + 1 + (1 - y) = 3 - y$. Because the perimeters are equal, it follows that $3x = 3y - x = 3 - y$. Solving this system yields $x = \frac{9}{13}$ and $y = \frac{12}{13}$. Thus $DE + FG = x + y = \frac{21}{13}$.

12. **Answer (A):** Let the angles of the triangle be $\alpha - \delta$, α, and $\alpha + \delta$. Then $3\alpha = \alpha - \delta + \alpha + \alpha + \delta = 180^\circ$, so $\alpha = 60^\circ$. There are three cases depending on which side is opposite to the 60° angle. Suppose that the triangle is ABC with $\angle BAC = 60^\circ$. Let D be the foot of the altitude from C. The triangle CAD is a 30-60-90° triangle, so $AD = \frac{1}{2}AC$ and $CD = \frac{\sqrt{3}}{2}AC$. There are three cases to consider. In each case the Pythagorean Theorem can be used to solve for the unknown side.

    If $AB = 5$, $AC = 4$, and $BC = x$, then $AD = 2$, $CD = 2\sqrt{3}$, and $BD = |AB - AD| = 3$. It follows that $x^2 = BC^2 = CD^2 + BD^2 = 21$, so $x = \sqrt{21}$.

If $AB = x$, $AC = 4$, and $BC = 5$, then $AD = 2$, $CD = 2\sqrt{3}$, and $BD = |AB - AD| = |x - 2|$. It follows that $25 = BC^2 = CD^2 + BD^2 = 12 + (x-2)^2$, and the positive solution is $x = 2 + \sqrt{13}$.

If $AB = x$, $AC = 5$, and $BC = 4$, then $AD = \frac{5}{2}$, $CD = \frac{5\sqrt{3}}{2}$, and $BD = |AB - AD| = |x - \frac{5}{2}|$. It follows that $16 = BC^2 = CD^2 + BD^2 = \frac{75}{4} + (x - \frac{5}{2})^2$, which has no solution because $\frac{75}{4} > 16$.

The sum of all possible side lengths is $2 + \sqrt{13} + \sqrt{21}$. The requested sum is $2 + 13 + 21 = 36$.

OR

As in the first solution, there are three cases depending on which side is opposite to the 60° angle. In each case, the Law of Cosines can be used to solve for the unknown side. If the unknown side is opposite to the 60° angle, then

$$x^2 = 4^2 + 5^2 - 2 \cdot 4 \cdot 5 \cdot \cos(60^\circ) = 21,$$

so $x = \sqrt{21}$.

If the side of length 5 is opposite to the 60° angle, then

$$5^2 = x^2 + 4^2 - 2 \cdot 4 \cdot x \cdot \cos(60^\circ) = x^2 - 4x + 16,$$

and the positive solution is $2 + \sqrt{13}$.

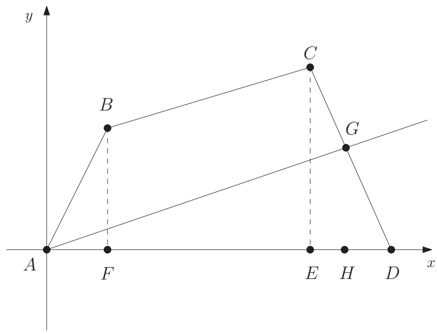
If the side of length 4 is opposite to the 60° angle, then

$$4^2 = x^2 + 5^2 - 2 \cdot x \cdot 5 \cdot \cos(60^\circ) = x^2 - 5x + 25,$$

which has no real solutions.

The sum of all possible side lengths is $2 + \sqrt{13} + \sqrt{21}$. The requested sum is $2 + 13 + 21 = 36$.

13. **Answer (B):** Let line AG be the required line, with G on $\overline{CD}$. Divide $ABCD$ into triangle ABF, trapezoid $BCEF$, and triangle CDE, as shown. Their areas are 1, 5, and $\frac{3}{2}$, respectively. Hence the area of $ABCD = \frac{15}{2}$, and the area of triangle $ADG = \frac{15}{4}$. Because $AD = 4$, it follows that $GH = \frac{15}{8} = \frac{r}{s}$. The equation of $\overline{CD}$ is $y = -3(x - 4)$, so when $y = \frac{15}{8}$, $x = \frac{p}{q} = \frac{27}{8}$. Therefore $p + q + r + s = 58$.



14. **Answer (B):**

Because the terms form an arithmetic sequence,

$$\begin{aligned}\log_{12} y &= \frac{1}{2}(\log_{12} 162 + \log_{12} 1250) = \frac{1}{2}\log_{12}(162 \cdot 1250)\\ &= \frac{1}{2}\log_{12}(2^2 3^4 5^4) = \log_{12}(2 \cdot 3^2 5^2).\end{aligned}$$

Then

$$\begin{aligned}\log_{12} x &= \frac{1}{2}(\log_{12} 162 + \log_{12} y) = \frac{1}{2}\left(\log_{12}(2 \cdot 3^4) + \log_{12}(2 \cdot 3^2 5^2)\right)\\ &= \frac{1}{2}\log_{12}(2^2 3^6 5^2) = \log_{12}(2 \cdot 3^3 5) = \log_{12} 270.\end{aligned}$$

Therefore $x = 270$.

OR

If $(B_k) = (\log_{12} A_k)$ is an arithmetic sequence with common difference d, then (A_k) is a geometric sequence with common ratio $r = 12^d$. Therefore $162, x, y, z, 1250$ is a geometric sequence. Let r be their common ratio. Then $1250 = 162r^4$ and $r = \frac{5}{3}$. Thus $x = 162r = 162 \cdot \frac{5}{3} = 270$.

15. **Answer (D):** There are two cases. If Peter and Pauline are given to the same pet store, then there are 4 ways to choose that store. Each of the children must then be assigned to one of the other three stores, and this can be done in $3^3 = 27$ ways. Therefore there are $4 \cdot 27 = 108$ possible assignments in this case. If Peter and Pauline are given to different stores, then there are $4 \cdot 3 = 12$ ways to choose those stores. In this case, each of the children must be assigned to one of the other two stores, and this can be done in $2^3 = 8$ ways. Therefore there are $12 \cdot 8 = 96$ possible assignments in this case. The total number of assignments is $108 + 96 = 204$.

16. **Answer (E):** Let a, b, and c be the number of rocks in piles A, B, and C, respectively. Then

$$\frac{40a + 50b}{a + b} = 43 \text{ and } 7b = 3a.$$

Because 7 and 3 are relatively prime, there is a positive integer k such that $a = 7k$ and $b = 3k$. Let μ_C equal the mean weight in pounds of the rocks in C and μ_{BC} equal the mean weight in pounds of the rocks in B and C. Then

$$\frac{40 \cdot 7k + \mu_C \cdot c}{7k + c} = 44, \text{ so } \mu_C = \frac{28k + 44c}{c},$$

and

$$\mu_{BC} = \frac{50 \cdot 3k + (28k + 44c)}{3k + c} = \frac{178k + 44c}{3k + c}.$$

Clearing the denominator and rearranging yields $(\mu_{BC} - 44)c = (178 - 3\mu_{BC})k$. Because the mean weight of the rocks in the combined piles A and C is 44 pounds, and the mean weight of the rocks in B is greater than the mean weight of the rocks in A, it follows that the mean weight of the rocks in B and C must be greater than 44 pounds. Thus $(\mu_{BC} - 44)c > 0$ and therefore $178 - 3\mu_{BC}$ must be greater than zero. This implies that $\mu_{BC} < \frac{178}{3} = 59\frac{1}{3}$. If $k = 15c$ and $\mu_C = 464$, then $\mu_{BC} = 59$. Thus the greatest possible integer value for the weight in pounds of the combined piles B and C is 59.

17. **Answer (D):** For $1 \le k \le 11$, the number of coins remaining in the chest before the k^{th} pirate takes a share is $\frac{12}{12-k}$ times the number remaining afterward. Thus if there are n coins left for the 12^{th} pirate to take, the number of coins originally in the chest is

$$\frac{12^{11} \cdot n}{11!} = \frac{2^{22} \cdot 3^{11} \cdot n}{2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11} = \frac{2^{14} \cdot 3^7 \cdot n}{5^2 \cdot 7 \cdot 11}.$$

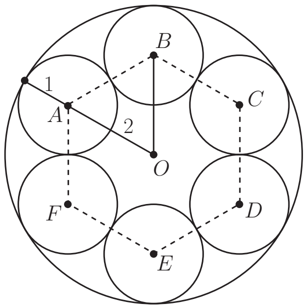
The smallest value of n for which this is a positive integer is $5^2 \cdot 7 \cdot 11 = 1925$. In this case there are

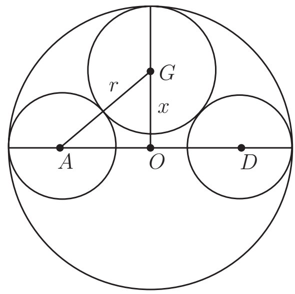
$$2^{14} \cdot 3^7 \cdot \frac{11!}{(12-k)! \cdot 12^{k-1}}$$

coins left for the k^{th} pirate to take, and note that this amount is an integer for each k. Hence the 12^{th} pirate receives 1925 coins.

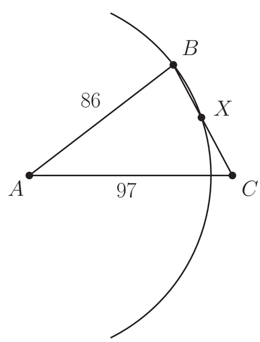
18. **Answer (B):** Let the vertices of the regular hexagon be labeled in order A, B, C, D, E, and F. Let O be the center of the hexagon, which is also the center of the largest sphere. Let the eighth sphere have center G and radius r. Because the centers of the six small spheres are each a distance 2 from O and the small spheres have radius 1, the radius of the largest sphere is 3. Because G

is equidistant from A and D, the segments $\overline{GO}$ and $\overline{AO}$ are perpendicular. Let x be the distance from G to O. Then $x + r = 3$. The Pythagorean Theorem applied to $\triangle AOG$ gives $(r+1)^2 = 2^2 + x^2 = 4 + (3-r)^2$, which simplifies to $2r + 1 = 13 - 6r$, so $r = \frac{3}{2}$. Note that this shows that the eighth sphere is tangent to $\overline{AD}$ at O.





19. **Answer (D):** By the Power of a Point Theorem, $BC \cdot CX = AC^2 - r^2$ where $r = AB$ is the radius of the circle. Thus $BC \cdot CX = 97^2 - 86^2 = 2013$. Since $BC = BX + CX$ and CX are both integers, they are complementary factors of 2013. Note that $2013 = 3 \cdot 11 \cdot 61$, and $CX < BC < AB + AC = 183$. Thus the only possibility is $CX = 33$ and $BC = 61$.



20. **Answer (B):** Consider the elements of S as integers modulo 19. Assume $a \succ b$. If $a > b$, then $a - b \le 9$. If $a < b$, then $b - a > 9$; that is $b - a \ge 10$ and so $(a + 19) - b \le 9$. Thus $a \succ b$ if and only if $0 < (a - b) \pmod{19} \le 9$.

Suppose that (x, y, z) is a triple in $S \times S \times S$ such that $x \succ y$, $y \succ z$, and $z \succ x$. There are 19 possibilities for the first entry x. Once x is chosen, y can equal

$x+i$ for any i, $1 \le i \le 9$. Then z is at most $x+9+i$ and at least $x+10$, so once y is chosen, there are i possibilities for the third element z.

The number of required triples is equal to $19(1+2+\cdots+9) = 19 \cdot \frac{1}{2} \cdot 9 \cdot 10 = 19 \cdot 45 = 855$.

21. **Answer (A):**

Let $A_n = \log\left(n + \log\left((n-1) + \log\left(\cdots + \log\left(3 + \log 2\right)\cdots\right)\right)\right)$. Note that $0 < \log 2 = A_2 < 1$. If $0 < A_{k-1} < 1$, then $k < k + A_{k-1} < k+1$. Hence $0 < \log k < \log(k + A_{k-1}) = A_k < \log(k+1) \le 1$, as long as $\log k > 0$ and $\log(k+1) \le 1$, which occurs when $2 \le k \le 9$. Thus $0 < A_n < 1$ for $2 \le n \le 9$.

Because $0 < A_9 < 1$, it follows that $10 < 10 + A_9 < 11$, and so $1 = \log(10) < \log(10 + A_9) = A_{10} < \log(11) < 2$. If $1 < A_{k-1} < 2$, then $k+1 < k + A_{k-1} < k+2$. Hence $1 < \log(k+1) < \log(k + A_{k-1}) = A_k < \log(k+2) \le 2$, as long as $\log(k+1) > 1$ and $\log(k+2) \le 2$, which occurs when $10 \le k \le 98$. Thus $1 < A_n < 2$ for $10 \le n \le 98$.

In a similar way, it can be proved that $2 < A_n < 3$ for $99 \le n \le 997$, and $3 < A_n < 4$ for $998 \le n \le 9996$.

For $n = 2012$, it follows that $3 < A_{2012} < 4$, so $2016 < 2013 + A_{2012} < 2017$ and $\log 2016 < A_{2013} < \log 2017$.

22. **Answer (E):** Let n be a 6-digit palindrome, $m = \frac{n}{11}$, and suppose m is a palindrome as well. First, if m is a 4-digit number, then $n = 11m < 11 \cdot 10^5 = 10^6 + 10^5$. Thus the first and last digit of n is 1. Thus the last digit of m is 1 and then the first digit of m must be 1 as well. Then $m \le 1991 < 2000$ and $n = 11m < 11 \cdot 2000 = 22\,000$, which is a contradiction. Therefore m is a 5-digit number $abcba$. If $a+b \le 9$ and $b+c \le 9$, then there are no carries in the sum $n = 11m = abcba0 + abcba$; thus the digits of n in order are a, $a+b$, $b+c$, $b+c$, $a+b$, and a. Conversely, if $a+b \ge 10$, then the first digit of n is $a+1$ and the last digit a; and if $a+b \le 9$ but $b+c \ge 10$, then the second digit of n is $a+b+1$ if $a+b < 9$, or 0 if $a+b = 9$, and the previous to last digit is $a+b$. In any case n is not a palindrome. Therefore $n = 11m$ is a palindrome if and only if $a+b \le 9$ and $b+c \le 9$.

Thus the number of pairs (m, n) is equal to

$$\sum_{b=0}^{9} \sum_{c=0}^{9-b} \sum_{a=1}^{9-b} 1 = \sum_{b=0}^{9} (10-b)(9-b).$$

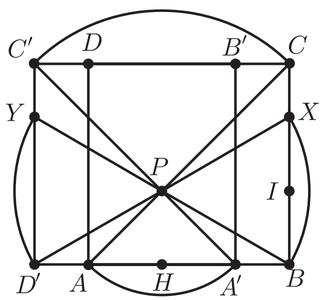
Letting $j = 10 - b$ gives

$$\sum_{j=1}^{10} j(j-1) = \frac{10 \cdot 11 \cdot 21}{6} - \frac{10 \cdot 11}{2} = 330.$$

The total number of 6-digit palindromes *abccba* is determined by 10 choices for each of b and c, and 9 choices for a, for a total of $9 \cdot 10^2 = 900$. Thus the required probability is $\frac{330}{900} = \frac{11}{30}$.

23. **Answer (C):**

Assume that the vertices of $ABCD$ are labeled in counterclockwise order. Let A', B', C', and D' be the images of A, B, C, and D, respectively, under the rotation. Because $\triangle A'PA$ and $\triangle C'PC$ are isosceles right triangles, points A' and C' are on lines AB and CD, respectively. Moreover, because $AP = \sqrt{2}$ and $PC = AC - AP = \sqrt{2}(\sqrt{3}+1) - \sqrt{2} = \sqrt{6}$, it follows that $AA' = \sqrt{2}AP = 2$ and $CC' = \sqrt{2}CP = 2\sqrt{3}$. By symmetry, points B' and D' are on lines CD and AB, respectively. Let $X \neq B$ and $Y \neq D'$ be the intersections of $\overline{BC}$ and $\overline{C'D'}$, respectively, with the circle centered at P with radius PB. Note that $PD' = PD = PB$, so this circle also contains D'. Therefore the required region consists of sectors APA', BPX, CPC', and YPD', and triangles BPA', CPX, YPC', and APD'.



Sector APA' has area $\frac{1}{4}\cdot(\sqrt{2})^2\pi = \frac{\pi}{2}$, and sector CPC' has area $\frac{1}{4}\cdot(\sqrt{6})^2\pi = \frac{3\pi}{2}$. Let H and I be the midpoints of $\overline{AA'}$ and $\overline{BX}$, respectively. Then $PH = AH = \frac{\sqrt{2}}{2}AP = 1$, and $PI = HB = AB - AH = \sqrt{3}$. Thus $\triangle BPH$ is a 30-60-90° triangle, implying that $PB = 2$ and $\triangle XPB$ is equilateral. Therefore congruent sectors BPX and YPD' each have area $\frac{1}{6}\cdot 2^2\pi = \frac{2\pi}{3}$.

Congruent triangles BPA' and $D'PA$ each have altitude $PH = 1$ and base $A'B = AB - AH - HA' = \sqrt{3}-1$, so each has area $\frac{1}{2}(\sqrt{3}-1)$. Congruent triangles CPX and $C'PY$ each have altitude $PI = \sqrt{3}$ and base $XC = BC - BX = \sqrt{3}-1$, so each has area $\frac{1}{2}(3-\sqrt{3})$.

The area of the entire region is

$$\frac{\pi}{2} + \frac{3\pi}{2} + 2\cdot\frac{2\pi}{3} + 2\left(\frac{\sqrt{3}-1}{2}\right) + 2\left(\frac{3-\sqrt{3}}{2}\right) = \frac{10\pi+6}{3},$$

and $a+b+c = 10+6+3 = 19$.

24. **Answer (E):** Assume without loss of generality that the regular 12-gon is inscribed in a circle of radius 1. Every segment with endpoints in the 12-gon subtends an angle of $\frac{360}{12}k = 30k$ degrees for some $1 \le k \le 6$. Let d_k be the length of those segments that subtend an angle of $30k$ degrees. There are 12 such segments of length d_k for every $1 \le k \le 5$ and 6 segments of length d_6. Because $d_k = 2\sin(15k^\circ)$, it follows that $d_2 = 2\sin(30^\circ) = 1$, $d_3 = 2\sin(45^\circ) = \sqrt{2}$, $d_4 = 2\sin(60^\circ) = \sqrt{3}$, $d_6 = 2\sin(90^\circ) = 2$,

$$\begin{aligned} d_1 &= 2\sin(15^\circ) = 2\sin(45^\circ - 30^\circ) \\ &= 2\sin(45^\circ)\cos(30^\circ) - 2\sin(30^\circ)\cos(45^\circ) = \frac{\sqrt{6}-\sqrt{2}}{2}, \text{ and} \\ d_5 &= 2\sin(75^\circ) = 2\sin(45^\circ + 30^\circ) \\ &= 2\sin(45^\circ)\cos(30^\circ) + 2\sin(30^\circ)\cos(45^\circ) = \frac{\sqrt{6}+\sqrt{2}}{2}. \end{aligned}$$

If $a \le b \le c$, then $d_a \le d_b \le d_c$ and the segments with lengths d_a, d_b, and d_c do not form a triangle with positive area if and only if $d_c \ge d_a + d_b$. Because $d_2 = 1 < \sqrt{6} - \sqrt{2} = 2d_1 < \sqrt{2} = d_3$, it follows that for $(a,b,c) \in \{(1,1,3),(1,1,4),(1,1,5),(1,1,6)\}$, the segments of lengths d_a, d_b, d_c do not form a triangle with positive area. Similarly,

$$\begin{aligned} d_3 &= \sqrt{2} < \frac{\sqrt{6}-\sqrt{2}}{2} + 1 = d_1 + d_2 < \sqrt{3} = d_4, \\ d_4 &< d_5 = \frac{\sqrt{6}+\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{2} + \sqrt{2} = d_1 + d_3, \text{ and} \\ d_5 &< d_6 = 2 = 1 + 1 = 2d_2, \end{aligned}$$

so for $(a,b,c) \in \{(1,2,4),(1,2,5),(1,2,6),(1,3,5),(1,3,6),(2,2,6)\}$, the segments of lengths d_a, d_b, d_c do not form a triangle with positive area. Finally, if $a \ge 2$ and $b \ge 3$, then $d_a + d_b \ge d_2 + d_3 = 1 + \sqrt{2} > 2 \ge d_c$, and also if $a \ge 3$, then $d_a + d_b \ge 2d_3 = 2\sqrt{2} > 2 = d_c$. Therefore the complete list of forbidden triples (d_a, d_b, d_c) is given by $(a,b,c) \in \{(1,1,3),(1,1,4),(1,1,5),(1,1,6),(1,2,4),(1,2,5),(1,2,6),(1,3,5),(1,3,6),(2,2,6)\}$.

For each $(a,b,c) \in \{(1,1,3),(1,1,4),(1,1,5)\}$, there are $\binom{12}{2}$ pairs of segments of length d_a and 12 segments of length d_c. For each $(a,b,c) \in \{(1,1,6),(2,2,6\}\}$, there are $\binom{12}{2}$ pairs of segments of length d_a and 6 segments of length d_c. For each $(a,b,c) \in \{(1,2,4),(1,2,5),(1,3,5)\}$, there are 12^3 triples of segments with lengths d_a, d_b, and d_c. Finally, for each $(a,b,c) \in \{(1,2,6),(1,3,6)\}$, there are 12^2 pairs of segments with lengths d_a and d_b, and 6 segments of length d_c. Because the total number of triples of segments equals $\binom{\binom{12}{2}}{3} = \binom{66}{3}$, the required probability equals

$$\begin{aligned} &1 - \frac{3 \cdot 12 \cdot \binom{12}{2} + 2 \cdot 6 \cdot \binom{12}{2} + 3 \cdot 12^3 + 2 \cdot 12^2 \cdot 6}{\binom{66}{3}} \\ &= 1 - \frac{63}{286} = \frac{223}{286}. \end{aligned}$$

25. **Answer (A):** Let $H = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. If $z_1, z_2 \in H$ and $f(z_1) = f(z_2)$, then $z_1^2 - z_2^2 + i(z_1 - z_2) = (z_1 - z_2)(z_1 + z_2 + i) = 0$. Because $\mathrm{Im}(z_1) > 0$ and $\mathrm{Im}(z_2) > 0$, it follows that $z_1 + z_2 + i \neq 0$. Thus $z_1 = z_2$; that is, the function f is one-to-one on H. Let r be a positive real number. Note that $f(r) = r^2 + 1 + ir$ describes the top part of the parabola $x = y^2 + 1$. Similarly, $f(-r) = r^2 + 1 - ir$ describes the bottom part of the parabola $x = y^2 + 1$. Because $f(i) = -1$, it follows that the image set $f(H)$ equals $\{w \in \mathbb{C} : \mathrm{Re}(w) < (\mathrm{Im}(w))^2 + 1\}$. Thus the set of complex numbers $w \in f(H)$ with integer real and imaginary parts of absolute value at most 10 is equal to

$$S = \{w = a + ib \in \mathbb{C} : a, b \in \mathbb{Z},\ |a| \le 10,\ |b| \le 10, \text{ and } a < b^2 + 1\}.$$

Because f is one-to-one, the required answer is $|f^{-1}(S)| = |S|$ and

$$1^2 - \sum_{b=-3}^{3} \sum_{a=b^2+1}^{10} 1 = 441 - \sum_{b=-3}^{3} (10 - b^2)$$

$$41 - (1 + 6 + 9 + 10 + 9 + 6 + 1) = 399.$$

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